CONSISTENT GRAVITATIONAL ANOMALIES FOR CHIRAL SCALARS

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Starting from the Henneaux-Teitelboim action for a chiral scalar, which generalizes to curved space the Floreanini-Jackiw action, we give two simple derivations of the exact consistent gravitational anomaly. The first derivation is through the Schwinger-DeWitt regularization. The second exploits cohomological methods and uses the fact that in dimension two the diffeomorphism transformations are described by a single ghost which allows to climb the cohomological chain in a unique way.

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1. Gravitational anomalies

The simplest instance of pure gravitational anomaly is the one due to the presence of a chiral fermion in two dimensions (Ref. 1). There are also gravitational anomalies produced by boson fields i.e. by the self-dual and anti self-dual fields which are realized in the simplest instance by the chiral scalars in dimension 2. The coupling of (anti) self-dual tensors to gravity is given by the Henneaux-Teitelboim action (Ref. 2) which generalizes to curved backgrounds the Floreanini-Jackiw action (Ref. 3) for chiral scalars in two dimensions. On curved background the chirality condition becomes (Ref. 4) ${E_+}^\mu \partial_\mu \varphi = 0$ where E_+^μ are the inverse zweibeins. In the following the key role will be played by the adimensional function

$$K = \frac{{E_+}^1}{{E_+}^0} = \frac{N}{\sqrt{h}} - N^1 = \frac{\sqrt{-g} - g_{01}}{g_{11}}. \tag{1}$$

The action is provided by (Ref. 2)

$$S = -\frac{1}{2}\int d^2x\ \partial_1\varphi(\partial_0\varphi + K\partial_1\varphi) = \frac{1}{2}\int d^2x\ \varphi\partial_1(\partial_0 + K\partial_1)\varphi. \tag{2}$$

The lack of explicit invariance under diffeomorphisms is the origin of the anomaly. The variation of S w.r.t φ gives the equation of motion

$$\partial_1(\partial_0 + K\partial_1)\varphi = 0 \tag{3}$$

and the action vanishes on the equation of motion. Under an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ the transformation of K and φ are $\delta_\xi K = -\partial_0\Xi - \partial_1\Xi\, K + \Xi\,\partial_1 K$ and $\delta_\xi\varphi = \Xi\partial_1\varphi$, with $\Xi = \xi^1 - K\xi^0$. Action (2) is invariant (Ref. 4) under such transformation. Thus only one combination of ξ^1, ξ^0 enters the transformation (Ref. 5).

2. The exact consistent anomaly through Schwinger DeWitt

The generating functional is given by

$$Z[K] = e^{iW[K]} = \int \mathcal{D}[\phi] \exp\left[i\frac{1}{2}\int d^2x\phi(\partial_1\partial_0 + \partial_1 K\partial_1)\phi\right] \equiv (\det(-iH))^{-\frac{1}{2}}\,. \quad (4)$$

The anomaly is provided by the variation of (4) under an infinitesimal diffeomorphisms. We have for the variation $i\delta_\xi W[K]$ the following expression

$$\int \mathcal{D}[\phi] e^{\frac{i}{2}\int d^2x\phi(\partial_1(K\partial_1+\partial_0)\phi}\int d^2x\frac{i}{2}\phi\partial_1(\delta_\xi K\partial_1\phi)/Z[K] = \frac{1}{2}\int d^2x\;\delta_\xi H\;G(x,x')|_{x'=x} \quad (5)$$

with $G(x,x')$ is the exact Green function in the external field K, which will be regularized à la Schwinger-DeWitt

$$G(x,x',\varepsilon) = i\langle x|\int_\varepsilon^\infty e^{iHt}dt|x'\rangle \quad \text{while} \quad \delta_\xi H = \partial_1(\Xi H) - H\Xi\partial_1. \quad (6)$$

Taking into account that H is the Laplace-Beltrami operator in the metric $g_{11} = 0,\; g_{10} = g_{01} = 2,\; g_{00} = -4K$ we obtain (Ref. 6) using the Seeley-DeWitt technique

$$\delta_\xi W = -\frac{1}{24\pi}\int d^2x\;\Xi(x)\;\partial_1^3 K(x) = \frac{1}{24\pi}\int d^2x\;K(x)\;\partial_1^3\Xi(x) \equiv G^E[K,\Xi] \quad (7)$$

which is the Einstein anomaly.

3. W-Z consistency condition and non triviality of the anomaly

Introducing the anticommuting diffeomorphism ghosts v^1, v^2 the BRST variation of K is

$$\delta K = -\partial_0 V - \partial_1 V K + V\partial_1 K \quad (8)$$

where $V = v^1 - Kv^0$, it is possible to show (Ref. 6) that (7) satisfies the Wess-Zumino consistency relation $\delta G^E[K,V] = 0$ and that the anomaly is not trivial. I.e. with $Q^1_2 = V\partial_1^3 K dx^1\wedge dx^0$ and $\delta Q^1_2 = -dQ^2_1$, $\delta Q^2_1 = -dQ^3_0 = -\frac{1}{2}d(V\partial_1 V\partial^2 V) \equiv -dN^3_0$, we have $Q^3_0 \neq \delta X^2_0$ for any X^2_0.

4. Cohomological derivation of the anomaly

Very general cohomological treatments of anomalies for conformal invariant theories have been given (Ref. 7,8) using two ghosts. Here exploiting the fact that in our case we can work with a single ghost (Ref. 5) the cohomological procedure can be streamlined. In the following discussion we shall denote by $S^n(m)$ the space of terms containing n ghosts and m derivatives, e.g. $V\partial_1^2 V\partial_0 Kf(K) \in S^2(3)$. The uniqueness of the last term Q^3_0 in the cohomological chain is equivalent to proving that the sequence

$$S^0(0) \xrightarrow{\delta} S^1(1) \xrightarrow{\delta} S^2(2) \xrightarrow{\delta} S^3(3) \xrightarrow{\delta} S^4(4) \xrightarrow{\delta} 0 \quad (9)$$

differs from an exact sequence only in the penultimate junction due to the presence of the non trivial term $N_0^3 \equiv \text{const}\ V\partial_1 V\partial_1^2 V$. A great simplification (Ref. 6) is obtained by performing a change of basis replacing the basis element $\partial_0 V$ by $W \equiv \delta K \equiv -\partial_0 V - \partial_1 V K + V\partial_1 K$, which is equivalent to it due to the relation (8). Then the algebra we need to use is simply $\delta V = V\partial_1 V;\ \delta K = W;\ \delta W = 0$. The sequence (9) is shown in Fig.1 where the numbers on the vertical bars denote the dimension of the space. The next step is to show the uniqueness of the term Q_1^2 up to trivial additions. To this end we have to prove that the kernel of δ from $S^2(3)$ into $S^3(4)$ is zero, modulo the trivial terms $\delta S^1(2)$. To this end we show (Ref. 6) that the sequence

$$0 \xrightarrow{\delta} S^0(1) \xrightarrow{\delta} S^1(2) \xrightarrow{\delta} S^2(3) \xrightarrow{\delta} S^3(4) \tag{10}$$

is exact.

We are left now with climbing the last step of the cohomology chain i.e. we have to prove the uniqueness, up to trivial terms, of the solution of

$$\delta Q_2^1 = -dQ_1^2 \tag{11}$$

of which we know already a solution i.e. $Q_2^1 = \text{const}\ V\partial_1^3 K dx^1 \wedge dx^0$. It corresponds to proving the exactness of sequence

$$0 \xrightarrow{\delta} S^0(2) \xrightarrow{\delta} S^1(3) \xrightarrow{\delta} S^2(4). \tag{12}$$

which can be easily performed using the algebra and the change of basis described after eq.(9).

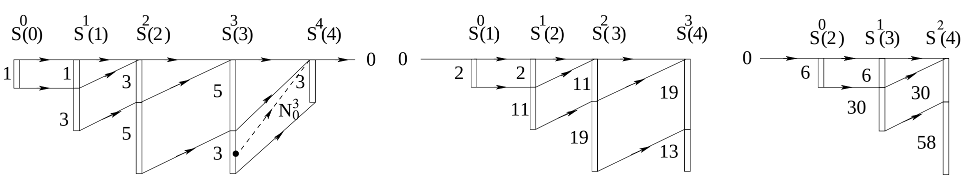

Fig. 1. The three cohomological sequences

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